THE
AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. III. APRIL, 1896. No. 4.

BIOGRAPHY.

JOHN NEWTON LYLE.

BY F. P. MATZ, SC. D., PH. D., PROFESSOR OF MATHEMATICS AND ASTRONOMY IN IRVING COLLEGE, MECHANICSBURG, PENNSYLVANIA.

JOHN NEWTON LYLE was born in Ralls County, Missouri, March 5, 1836.

"The space in which this county is located is trinally extended, and therefore objective. It has no curvature, either positive or negative. Here planes are flat, and perpendiculars to a transversal are equidistant.

If Lobatschewsky had been born and raised in Ralls County, he would perhaps never have doubted *that two straight lines equidistant from each other may be drawn in the same plane*, nor written a theory of parallels in which this postulate of sound geometry is discredited. The hills, rocks, streams, trees, and hard-pan of Ralls County exist in tridimensional space,—objective to the minds of the inhabitants who till the soil, feed the herds, quarry the rock, fell the trees, and hunt the wild game. These physical objects are realities having an *objective* existence ; and the space occupied by them, and that in which they are contained, is also an objective *entity* although not a *material* one. It is believed that Helmholtz was right in maintaining against Kant the objectivity of space, but wrong in regarding it as a *physical thing* to be moulded like potter's clay. Sir Isaac Newton held the opinion that space was immaterial, immovable, and unalterable, as well as trinally extended, continuous, and unbounded.

Immanuel Kant professed to cognize a real, objective, *extended* world as existing in *a space that*, according to his philosophy *had no existence outside of his own mind*. It would seem that if there is an extended world, there must be

JOHN NEWTON LYLE.

an extended spatial entity to contain it. If space is not *extended*, and therefore not objective, there can be no real, extended, objective, material world.

If Immanuel Kant's experiences in early life had been those of a pioneer's son in Ralls County, Missouri, he would not in all probability have undertaken in his riper years the contract of building a *real world* in a *non-existent* place.

If Fichte, or Hegel, had ever galloped after a wild steer for half a day through a Ralls County forest or been thrown from a bucking mustang on the phenomenal hard-pan of northeastern Missouri, they doubtless would not have felt inclined to regard this real matter-of-fact world as *an idealistic dream*."

The ancestors of John Newton Lyle, on the paternal side, came from northern Ireland ; and on the maternal side, from England and Wales. They settled in Berkeley County, Virginia, in the last century ; and many of their descendants are still to be found there.

Samuel Oldham Lyle, the father of the subject of this sketch, emigrated from Berkeley County, Virginia, in 1832, to Ralls County, Missouri, where he purchased a farm ; married, Ann Rebecca, the daughter of William Gerard, and reared his family. This pioneer couple were intellectual in their tastes, great readers, ambitious to make a pleasant home for their children, and give each of them an education.

Ann Rebecca's father, William Gerard, emigrated from Berkeley County, Virginia, to Kentucky, during the last decade of the last century, where he learned the printing business, and edited and published for many years the *Argus*, a newspaper, at Frankfort, Kentucky. He was a man of affairs, as well as of extensive reading ; and he was also a practical politician intimately associated with the statesmen of his adopted State. He came with his family to northeastern Missouri, in 1830.

John Newton Lyle, in his early boyhood, was thrown in his Grandfather Gerard's society a great deal, and received from him powerful impulses towards intellectual pursuits. The venerable man treated his grandson more as a companion than as a boy needing a rod for his misdemeanors, aroused his curiosity by well-directed questions, corrected his mistakes, and entertained him with anecdotes about Amos Kendall, the elder Blair, and Henry Clay.

Samuel Oldham Lyle was an enterprising, independent, and fearless pioneer, passionately fond of the chase and free life in the wild west ; but, at the same time, he was diligent in his farming and stock-raising. He was a man of quick intelligence, unfailing memory, and sound judgment, who appreciated the value and importance of education ; and he gave to his children the best school advantages that his circumstances would allow.

Young Lyle, at six years of age, was placed in a district school ; and here he remained until he reached the age of twelve. In November, 1848, he entered a classical school taught by the Rev. William T. Dickson, at West Ely, Marion County, three miles from the home of the young pupil. He studied the rudiments of the Latin and Greek languages, Euclid's Elements, and Day's Algebra.

Mr. Dickson was a native of the State of Maine, and came West with Dr.

Ezra Styles Ely, to attend Marion College, some time in the Thirties. "He was an enthusiastic and successful instructor in the branches of learning that he professed to teach. He did not tell his scholars anything about differential coefficients, integrals, or Cartesian co-ordinates. He was silent as to determinants, trilinears, and Non-Euclidean Geometry. He did understand Euclid's Elements, however ; and he taught the science, clearly, thoroughly, and ably. With him, straight lines were never *flexed* or *curved*. Tangents to circumferences were never confounded with the curves to which they were tangent. Planes were flat superficies ; and, in no instance, were they spherical or pseudo-spherical. He showed the *meaning* of demonstration, by *demonstrating* theorems. He illustrated by practical examples, the difference between direct and indirect proof. He was a true teacher, and succeeded well in imparting to his pupils something of his own appreciation and admiration of the enduring work of Alexandria's immortal geometer."

In the fall of 1851, within three miles of Samuel Oldham Lyle's farm, Van Rensselaer Academy was founded, at the head of which was the Rev. J. P. Finley, afterwards a professor in Westminster College, and the founder of a classical institution at Brookfield, Missouri. John Newton Lyle entered this Academy, in October, 1851 ; and he was a student there three successive winters, farming during the summer. Here he continued his studies in Latin and Greek, reviewed Euclid, then took up Davies' Legendre and Robinson's Algebra. At this time, he, also, studied Trigonometry and Surveying. Mr. Finley's tastes were classical, rather than mathematical ; and his pupil, J. N. Lyle, whilst at the Academy, devoted his energies almost exclusively to mastering the Latin and Greek texts put into his hands.

Before he was nineteen years of age he took charge of his first school in Monroe County, early in September, 1854 ; thus he began his long career as a teacher, which he has continued almost uninterruptedly until the present time. He worked with the definite plan of preparing for College and earning the funds necessary for securing a collegiate education. He taught two years in the public schools of Monroe County, spending his evenings and Saturdays in study.

During these years he plodded without assistance, through Davies' Analytical Geometry. Having finished this self-imposed task, a strong desire took possession of him to advance farther and investigate Davies' Differential and Integral Calculus. Accordingly one sultry day in August, 1856, he rode from his father's farm to Hannibal, purchased the book, and on returning home immediately sought a secluded spot in the forest and began the study of the first differential coefficient as explained by Charles Davies. He was *thoroughly disgusted* that hot August afternoon, with Davies' description of differentials as the "traces" of vanishing increments. He persevered, however, notwithstanding his dissatisfaction with the author's theory of differentials and differential coefficients. A copy of Loomis's Calculus, which came in his way, was eagerly studied. Loomis's theory of differentials as *rates of variation* had the advantage of being intelligible, and certainly offered something more substantial to be grasped by

the mind than a mere "trace" of a vanishing increment or the "ghost of a departed quantity."

"Rates of variation are *finite* quantities. If differentials are rates of variation, then, of necessity, they must be definite quantities. The Leibnitzian hypothhesis that differentials are *infinitely small quantities* contradicts the hypothesis that they are rates of variation." During the fall of 1856, he studied both Loomis and Davies on the Calculus. This work was done entirely without the instruction of a teacher ; because there was no one within reach, who had studied these branches, to whom he could apply for aid. "*This method of study*, whilst laborious and beset with many inconveniences, *was conducive to independence of thought and action, and the formation of the habit of self-reliance.*"

The first part of the year 1857, John Newton Lyle taught mathematics in Bethel College, a Baptist Institution located at Palmyra, Missouri. The apportunity of attending Marietta College, for which he had long planned and toiled, now presented itself. On examination he entered the Junior Class in Marietta College, the fall of 1857 ; and he continued in that Institution, until his graduation in 1859.

President Israel Ward Andrews conducted the examination in Mathematics, and expressed himself as highly gratified with the candidate's proficiency ; and on making inquiry as to who taught him Analytical Geometry, seemed amused when informed that his only instructor was the youthful pedagogue before him seeking admittance to the privileges of the College. Dr. Andrews was his warm and steadfast friend, from the date of that morning's interview on Mathematics.

J. N. Lyle, in College, sought to utilize the advantages of the library and his literary society as well as those of the recitation-room and the laboratory. His special delight was to participate in the Saturday-morning debates held in the hall of the Alpha Kappa Society. The enjoyableness of the excitement far outweighed the unpleasantness of the collisions incident to such exercises.

He lost no time in obtaining from the College Library De Morgan's Differential and Integral Calculus, in order to learn that author's opinions respecting the principles of the science. "He was interested in noting that De Morgan employed variables that increased, and decreased, indefinitely without limit, instead of the hierarchy of infinitely great, and infinitely small, quantities of the Leibnitzian hypothesis. Whilst no lost value was attributed to these variables, every value that they did have, was *finite*. The hypothesis of increasing, and decreasing, variables having finite values not only works well in practice, but has the advantage over the hypothesis of Leibnitz in that it is intelligible and does not involve contradiction. It also harmonizes well with the view that differentials are rates of variation. Further, in considering a limit, we note that the interval between a limit and the variable that approaches it, is itself a variable that decreases without limit. From this point of view, the absurdity of regarding a variable that increases without limit as having a limit appropriately symbolized by ∞, is quite evident.

No benefit accrues to the Science of the Calculus, from De Morgan's hypothesis that there are *two* kinds of zeros—*the absolute zero*, and *the indifinitely small quotient*. The absolute zero is destitute of all value ; in fact, it is the negation of quantity,—and hence can not be treated as quantity, without violating the logical law of Non-Contradiction. A quotient may become indefinitely small, but can not become so small as not to be quantity. To name a quotient zero, is manifestly a misnomer. Mathematical and logical confusion is liable to result from the ambiguous use of the symbol 0. Treating quantity as no quantity, or no quantity as quantity, is a procedure which may be profitably dispensed with in Mathematics."

E. W. Evans, of Yale, came to Marietta College, as Professor of Mathematics, at the same time that J. N. Lyle entered as a student. The young professor seemed very lonesome as his wife remained in the East that fall. He would come over to Lyle's room of evenings and remain for hours. His conversation which took a wide range was quite instructive to his western pupil. Mathematics was discussed a great deal, but not exclusively. He believed most religiously that "brevity is the soul of wit." He once said : "Lyle, the longer I live the more I like 'short things'." His pupil furnished his share of the intellectual picnic with anecdotes and experiences respecting that portion of the West where Mark Twain was born, Tom Sawyer flourished, and Captain Sellers bored with a big auger.

The two years immediately after graduation he spent in teaching in Pettis and Morgan Counties, Missouri. His leisure hours were occupied in reading law-books. In the spring of 1862, he was offered the chair of Mathematics and Natural Science in Westminster College, a position he held until 1865, when he went to Carondelet, a suburb of St. Louis, where he taught a Grammar School ; but in the fall of the same year, he accepted the position of Acting Professor of Mathematics and Natural Science in his *Alma Mater*, Marietta College. He continued there three years ; at the expiration of which time, he returned to Fulton, as Professor of Natural Science in Westminster College. Here he has since remained. First and last, as the exigencies of College-work might require, he has taught branches in nearly every department of the Institution.

He is an active member of *The Missouri Teacher's Academy*. To educational journals he has contributed hundreds of articles principally on Mathematical Philosophy. During the last three years preceding 1890, he had published in the *Missouri School Journal* not less than sixty-one articles. He has written an unpublished manuscript on the Differential and Integral Calculus.

The degree of Ph. D., was conferred on him by Marietta College, in 1881. In 1868, Professor Lyle was married to Miss Margaret T. Hays, daughter of John B. Hays, M. D., of Marion County, Missouri, who until her death, December 26, 1882, in spite of ill health and great suffering, led such a life of unselfish devotion to husband, children, and friends, as called forth constant admiration of the talent, energy, and piety, that enabled her to accomplish so much. Three of the five children of this couple are living, two daughters and a son, Rev. Edward

Hays Lyle, an alumnus of Westminster College, a Theological student of Princeton Seminary for two years, and at present a minister in charge of a church at La Junta, Colorado.

Dr. Lyle, in 1884, married his second wife, Miss Mattie E. Grant, a scholarly and cultured lady, of Bardstown, Kentucky.

Dr. Lyle has been for many years an Elder in the Presbyterian Church, the church of his ancestors for, at least, the century and a half that have elapsed since his Great Grandfather emigrated from the northern part of Ireland to Berkeley County, Virginia.

THE CENTROID OF AREAS AND VOLUMES.

By G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

[Concluded.]

We will now find the centroid of the eighth part of the surface

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1,\ \text{I, when } c=b,\ \text{II, } c=a.$$

We have $\bar{x}=\dfrac{\int xds}{\int ds}$, $\bar{y}=\dfrac{\int yds}{\int ds}$, $\bar{z}=\dfrac{\int zds}{\int ds}$.

$$\text{I.}\quad s=\frac{b}{a}\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\left\{\frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}dxdy$$

$$=\frac{\pi b}{2a^2}\int_0^a\sqrt{a^4-(a^2-b^2)x^2}\,dx=\tfrac{1}{4}\pi b\left(b+\frac{a}{e}\sin^{-1}e\right).$$

$$s.\bar{x}=\int xds=\frac{b}{a}\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\left\{\frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}xdxdy$$

$$= \frac{\pi b}{2a^2}\int_o^a \sqrt{a^4-(a^2-b^2)x^2}\, xdx = \frac{\pi ab(a^2+ab+b^2)}{6(a+b)}$$

$$\therefore \bar{x} = \frac{2a(a^2+ab+b^2)}{3(a+b)(b+\frac{a}{e}\sin^{-1}e)}.$$

$$s.\bar{y} = s.\bar{z} = \int yds = \frac{b}{a}\int_o^a\int_o^{\frac{b}{a}\sqrt{a^2-x^2}} \left\{ \frac{a^4-(a^2-b^2)x^2}{a^2b^2-b^2x^2-a^2y^2} \right\}^{\frac{1}{2}} ydxdy$$

$$= \frac{b^2}{a^2}\int_o^a \sqrt{(a^2-x^2)(a^2-e^2x^2)}dx = ab^2\int_o^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\cos^2\theta d\theta,\ x=a\sin\theta$$

$$= \frac{ab^2}{3e^2}\left\{ (1+e^2)\, E(e,\frac{\pi}{2}) - (1-e^2)F(e,\frac{\pi}{2}) \right\}.$$

$$\therefore \bar{y} = \bar{z} = \frac{4ab \left\{(1+e^2)E(e,\frac{\pi}{2}) - (1-e^2)F(e,\frac{\pi}{2})\right\}}{3\pi e^2(b+\frac{a}{e}\sin^{-1}e)}.$$

II. $$s = \frac{a}{b}\int_o^b\int_o^{\frac{a}{b}\sqrt{b^2-y^2}} \left\{ \frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2} \right\}^{\frac{1}{2}} dydx$$

$$= \frac{\pi a}{2b^2}\int_o^b \sqrt{b^4+(a^2-b^2)y^2}\,dy = \frac{\pi a^2}{4}\left\{ 1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right\}.$$

$$s.\bar{x} = s.\bar{z} = \int xds = \frac{a}{b}\int_o^b\int_o^{\frac{a}{b}\sqrt{b^2-y^2}} \left\{ \frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2} \right\}^{\frac{1}{2}} xdydx.$$

$$s.\bar{x} = s.\bar{z} = \frac{a^2}{b^3}\int_o^b \sqrt{(b^2-y^2)(b^4+a^2e^2y^2)}\,dy$$

$$= a^2\int_o^{\frac{1}{2}\pi} \sqrt{b^2+a^2e^2\cos^2\theta}\,\sin^2\theta d\theta,\ y=b\cos\theta$$

$$= a^3\int_o^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\,\sin^2\theta d\theta$$

$$=\frac{a^3}{3e^2}\left\{(1-e^2)F(e,\frac{\pi}{2})-(1-2e^2)E(e,\frac{\pi}{2})\right\}.$$

$$\therefore\ \bar{x}=\bar{z}=\frac{4a\left\{(1-e^2)F(e,\frac{\pi}{2})-(1-2e^2)E(e,\frac{\pi}{2})\right\}}{3\pi e^2(1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e})}.$$

$$s.\bar{y}=\frac{a}{b}\int_o^b\int_o^{\frac{a}{b}\sqrt{b^2-y^2}}\left\{\frac{b^4+(a^2-b^2)y^2}{a^2b^2-b^2x^2-a^2y^2}\right\}^{\frac{1}{2}}ydydx=\int yds$$

$$=\frac{\pi a}{2b^2}\int_o^b\sqrt{b^4+a^2e^2y^2}\,ydy=\frac{\pi ab(a^2+ab+b^2)}{6(a+b)}.$$

$$\therefore\ \bar{y}=\frac{2b(a^2+ab+b^2)}{3a(a+b)(1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e})}.$$

Since the limit of $\frac{\sin^{-1}e}{e}$ and $\frac{\log\frac{1+e}{1-e}}{2e}$ is 1 when $e=0$ we have, in either case, when $a=b$, $\bar{x}=\bar{y}=\bar{z}=\frac{1}{2}a$. The surface of the fourth part of the paraboloid $x^2+y^2=2a^2z$, for $z=h$.

$$s=\iint\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dz}\right)^2}\,dzdx=\iint\sqrt{1+\frac{x^2}{y^2}+\frac{a^4}{y^2}}\,dxdz.$$

$$\therefore\ s=a\int_o^h\int_o^{a\sqrt{2z}}\sqrt{\frac{a^2+2z}{2a^2z-x^2}}\,dzdx=\frac{\pi a}{2}\int_o^h\sqrt{a^2+2z}\,dz$$

$$=\frac{\pi a}{6}\left\{(a^2+2h)^{\frac{3}{2}}-a^3\right\}.$$

$$s.\bar{x}=s.\bar{y}=\int yds=a\int_o^h\int_o^{a\sqrt{2z}}\sqrt{a^2+2z}\,dzdx=a^2\int_o^h\sqrt{(a^2+2z)2z}\,dz$$

$$=\frac{a^2}{16}\left\{2(a^2+4h)\sqrt{2a^2h+4h^2}-a^4\log\left(\frac{a^2+4h+\sqrt{2a^2h+4h^2}}{a^2}\right)\right\}.$$

$$\therefore \bar{x}=\bar{y}=\frac{3a\left\{2(a^2+4h)\sqrt{2a^2h+4h^2}-a^4\log\left(\frac{a^2+4h+2\sqrt{2a^2h+4h^2}}{a^2}\right)\right\}}{8\pi\{(a^2+2h)^{\frac{3}{2}}-a^3\}}.$$

$$s.\bar{z}=\int zds=a\int_o^h\int_o^{a\sqrt{2z}}\sqrt{\frac{a^2+2z}{2a^2z-x^2}}zdzbx$$

$$=\frac{\pi a}{2}\int_o^h\sqrt{a^2+2z}\,zdz=\frac{\pi a}{30}\left\{(3h-a^2)(a^2+2h)^{\frac{3}{2}}+a^5\right\}.$$

$$\therefore \bar{z}=\frac{(3h-a^2)(a^2+2h)^{\frac{3}{2}}+a^5}{5\{(a^2+2h)^{\frac{3}{2}}-a^3\}}.$$

The surface of the fourth part of the cone $x^2+y^2=a^2z^2$, for $z=h$.

$$s=\int\int\sqrt{1+\frac{x^2}{y^2}+\frac{a^4z^2}{y^2}}\,dzdx=a\sqrt{1+a^2}\int_o^h\int_o^{az}\frac{zdzdx}{\sqrt{a^2z^2-x^2}}$$

$$=\frac{\pi a\sqrt{1+a^2}}{2}\int_o^h zdz=\frac{\pi ah^2\sqrt{1+a^2}}{4}.$$

$$s.\bar{x}=s.\bar{y}=\int yds=a\sqrt{1+a^2}\int_o^h\int_o^{az}zdzdx=a^2\sqrt{1+a^2}\int_o^h z^2dx=\frac{a^2h^3\sqrt{1+a^2}}{3}.$$

$$\therefore \bar{x}=\bar{y}=\frac{4ah}{3\pi}.$$

$$s.\bar{z}=\int zds=a\sqrt{1+a^2}\int_o^h\int_o^{az}\frac{z^2dzdx}{\sqrt{a^2z^2-x^2}}=\frac{\pi a\sqrt{1+a^2}}{2}\int_o^h z^2dz=\frac{\pi ah^3\sqrt{1+a^2}}{6}.$$

$$\therefore \bar{z}=\frac{2h}{3}.$$

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from March Number.]

Having disposed of the regular primitive groups we turn now to those whose order exceeds their degree. We have proved that all of these involve n conjugate subgroups whose degree is at most equal to $n-1$. Suppose the degree of these subgroups were $n-2$. Without loss of generality we may then assume that the following identities are satisfied :

$$G_1 \equiv G_2,\ G_3 \equiv G_4,\ \ldots\ldots\ G_{n-1} \equiv G_n.$$

If g_1 represents the order of G_1 we see that $2g_1$ substitutions of G transform G_1 into itself, viz., those which replace a_1 by itself and those which replace a_1 by a_2. All of the g_1 substitutions which replace a_1 by a_2 must therefore also replace a_2 by a_1, *i. e.*, contain the cycle a_1a_2. Similar remarks apply to the other couples a_3a_4, $\ldots\ldots a_{n-1}a_n$.

We inquire whether these couples may be used as systems of non-primitivity. We have already proved that every substitution that replaces one letter of a couple by the other contains the couple as a distinct cycle. It remains to show that the couples are interchanged as units by the substitutions of G. Suppose one of these substitutions t replaces a_1 by a_3. Then will

$$t\,G_3 t^{-1} = G_1$$

t must therefore replace a_2 by a_4. Since similar remarks apply to the other couples we have proved that the couples can be used as systems of non-primitivity.

In an exactly similar way we can prove the general case that if the degree of the conjugate subgroups is $n-\alpha$ $(\alpha > 1)$ then systems of α letters each may be used as systems of non-primitivity. Hence the

THEOREM. *Whenever a transitive group contains a subgroup whose degree is less than $n-1$ and which involves all the substitutions that do not contain a given letter it must be non-primitive.*

Having developed some of the most important properties of the subgroups of G which do not contain a given letter we proceed to inquire into their substitutions. Suppose that among the substitutions of a transitive group G

$$a_1 a_\alpha$$

has only one solution ; *i. e.*, there is only one cycle of this type in the group which contains a_1. Then there can be only one value of γ for each β in

$$a_\beta \, a_\gamma \qquad (\beta=1,\ 2,\ \ldots\ldots n)$$

since any a can be transformed into a_1. All the conjugates of $a_1 a_\alpha$ are therefore distinct and may be used as systems of non-primitivity of the given transitive group.

More generally speaking we may say that if G contains a subgroup G' whose degree n' is less than the degree of G and if any given letter of G (a_1) is found in only one of the transforms of G' with respect to G, then will these transforms

$$G',\ G'',\ \ldots\ldots,\ G^n$$

constitute systems of non-primitivity of G.

For if G^α and G^β had a common letter then would the substitution of G which transforms this common letter into a_1 lead to two such groups both of which would involve a_1. This is contrary to the hypothesis. These conjugate subgroups must therefore involve distinct sets of letters which may be regarded the systems of non-primitivity of G. Hence the

THEOREM. *If a primitive group contains a subgroup whose degree is less than the degree of the group it must also contain a substitution which transforms this subgroup into one which contains any one of its letters together with at least one new letter.*

From this theorem it follows that if a primitive group whose degree exceeds 2 contains the cycle $a_1 a_2$ it must also contain $a_1 a_3$ (a_3 representing any suitable letter, different from a_1 and a_2) and therefore the symmetric group of these three letters $(a_1 a_2 a_3)$all.

If a primitive group whose degree exceeds three contains $(a_1 a_2 a_3)$all it must, according to the given theorem, also contain $(a_1 a_\alpha a_\beta)$all where at least one of the two subscripts α, β exceeds 3. Representing this by 4 we can easily show that the group must contain at least all the substitutions of

$$(a_1 a_2 a_3 a_4)\text{all}$$

whose degree does not exceed 3. For if any such substitution is given we can find some substitution of $(a_1 a_2 a_3)$all which is either the same or differs from it only in having another letter a_α where the given substitution has a_4. The transform of this substitution with respect to $a_\alpha a_4$ (which is known to be in the group) will be the given substitution. Since every substitution of the fourth degree is the product of two substitutions of a lower degree the given primitive group must contain

$$(a_1 a_2 a_3 a_4)\text{all}.$$

In general, if a primitive group whose degree exceeds m contains

$$(a_1 a_2 \ldots\ldots a_m)\text{all}$$

it must also contain

$$(a_1 a_\alpha \ldots\ldots a_\mu)\text{all}$$

where the number of subscripts 1, α, ...μ is m and at least one of them exceeds m. Representing this by $m+1$ we see that G must contain

$$a_\alpha\, a_{m+1} \qquad (\alpha=1,\, 2,\, \ldots\ldots a_m).$$

We consider now any substitution of

$$(a_1 a_2 \ldots\ldots a_{m+1})\text{all}$$

whose degree does not exceed m. We can find some substitutions in

$$(a_1 a_2 \ldots\ldots a_m)\text{all}$$

which is either the same or differs from it only in having a_α where this has a_{m+1}. In this case the transform with respect to $a_\alpha\, a_{m+1}$ will be the given substitution. Since a substitution of the $m+1$ degree ($m \gtreqless 2$) may be regarded as the product of two substitutions of a lower degree the given primitive group must contain

$$(a_1 a_2 \ldots\ldots a_{m+1})\text{all}.$$

Calling $m+1$ m' we can prove in the same way that G contains the symmetric group of $m'+1=m+2$ letters, etc. Hence the

THEOREM. *Whenever a primitive group contains a symmetric subgroup of a lower degree it must be a symmetric group.*

COROLLARY. *If a primitive group contains a substitution of the form* $a_1 a_2$ *it is symmetric.*

We will now suppose that the primitive group contains

$$a_1 a_2 a_3.$$

If its degree exceeds 3 it must also contain

$$a_1 a_\alpha\, a_\beta$$

where at least one of the two letters, say α, is greater than 3. We shall represent this by 4, G then contains the two substitutions

$$a_1 a_2 a_3 \text{ and } a_1 a_4 a_\beta$$

and therefore

$$(a_1 a_2 a_3 a_4)\text{pos}.$$

In general, if a primitive group whose degree exceeds m contains

$$(a_1 a_2 \ldots\ldots a_m)\text{pos}$$

it must also contain

$$(a_1 a_\alpha \ldots\ldots a_\mu)\text{pos}$$

where the number of subscripts $1, \alpha, \ldots\ldots \mu$ is m and at least one of them exceeds m. Representing this by $m+1$ we see that G contains

$$a_\alpha\, a_{m+1} a_\beta \qquad (\alpha = 1, 2, \ldots\ldots m).$$

It must therefore contain at least all of the substitutions of

$$(a_1 a_2 \ldots\ldots a_{m+1})\text{pos}$$

whose degree does not exceed m. For if s is any such substitution containing a_{m+1} there is some substitution s_1 in

$$(a_1 a_2 \ldots\ldots a_m)\text{pos}$$

which differs from s only in having a_δ where s has a_{m+1}. If β exceeds m we make $\alpha = \delta$ then will $a_\alpha\, a_{m+1} a_\beta$ transform s_1 into s. If $\beta \overline{\gtrless} m$ we transform the substitution

$$a_\alpha\, a_{m+1} a_\beta$$

with respect to some substitution of $(a_1 a_2 \ldots\ldots a_m)$pos. So that in place of a_β we may have a letter not found in s. Let this transform be

$$a_\gamma\, a_{m+1} a_\varepsilon \qquad (\gamma,\ \varepsilon \overline{\gtrless} m).$$

We now take from the substitutions of $(a_1 a_2 \ldots\ldots a_m)$pos the one s_2 which differs from s only in having a_γ where s has a_{m+1} if s does not contain a_γ, and the one s_3 which differs from s only in having a_ε, a_γ where s has a_γ, a_{m+1} if s contains a_γ. The transform of these with respect to

$$a_\gamma\, a_{m+1} a_\varepsilon$$

will be the required substitution s.

This proves that G contains all the substitutions of $(a_1 a_2 \ldots\ldots a_{m+1})$pos whose degree is equal to or less than m. These generate $(a_1 a_2 \ldots\ldots a_{m+1})$pos, for any positive substitution of the $(m+1)^{th}$ degree $(m>2)$ may be considered as the product of two positive substitutions of a lower degree. [Let $s = \ldots a_x a_y \ldots$ be any positive substitution of the $(m+1)^{th}$ degree and $s_1 \ldots\ldots a_x a_y \ldots\ldots$ be any positive substitution of a lower than the $(m+1)^{th}$ degree. Then will s_2 in

$$s = s_1 s_2 \text{ or } s_2 = s_1^{-1} s$$

be also a positive substitution whose degree $\overline{\gtrless}m$].* Hence the

THEOREM. *If a primitive group contains a substitution of the form* $a_1a_2a_3$ *but none of the form* a_1a_2 *it is the alternating group.*

We are now in possession of the following important facts in regard to any primitive group G.

(1) If $g=n$, G must be generated by a single cycle which involves a prime number of letters, and for each prime number there is one and only one such primitive group.

(2) If g does not equal n it must be a larger multiple of n and G must contain n conjugate subgroups whose degree is $n-1$ and whose order is $g\div n$.

(3) If G contains a substitution of the form a_1a_2 or one of the form $a_1a_2a_3$ it must contain the alternating group.

(4) Both the alternating and the symmetric groups have a 1, 1 correspondence to the positive integers beginning with 2.

(5) The order of the symmetric group is n ! and that of the alternating group is $\frac{1}{2}n$!.

(6) The average number of letters in all the substitutions of a transitive group is $n-1$.

(7) Every transitive group contains at least $n-1$ substitutions of the n^{th} degree.

The three classes of primitive groups, regular, alternating, and symmetric, each of which contains an infinite number of members, are distinct when $n>3$. The groups that belong to these classes for any value of n are well known. It remains to determine those whose order satisfies the inequality

$$n>g>\tfrac{1}{2}n\,!.$$

Before pursuing the general discussion any farther we shall seek all the primitive groups whose degree does not exceed six. In doing this we shall use some methods which will be of service in the further study of this subject. Most of the methods, however, may serve as illustrations of the theorems which have been developed.

[To be Continued.]

---

*It can be easily proved that if a group contains

$$a_1a_2a_\alpha \qquad (\alpha=1,\ 2,\ \ldots\ldots n)$$

it contains the alternating group of degree n, and if it contains

$$a_1a_\alpha \qquad (\alpha=1,\ 2,\ \ldots\ldots n)$$

it contains the symmetric group of degree n. Cole's Netto, §§ 34, 35.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from March Number.]

Corollary II. But again I am able hence to show that those two straights AX, BX, meeting which the straight $PFHD$ makes either two internal angles toward the same parts equal to two right angles, or consequently (from Eu. I. 13 and 15) alternate external or internal angles equal to one another, or again, from the same cause, an external (as suppose DHX) equal to an internal and opposite HFX; that, say I, those two straights not even in their infinite production can meet one another.

For if from any point N of AX is let fall to BX the perpendicular NR, this will be in the hypothesis of acute angle (which alone in any case can hinder us) greater (from III. Cor. I.) than the common perpendicular KL. Therefore those two straights AX, BX cannot ever meet one another.

But furthermore here thou hast demonstrated propositions 27 and 28 of the first book of Euclid, and indeed without immediate dependence from the preceding 16 and 17 of the same first book, about which difficulties could arise when the triangle should be of infinite sides on a finite base; to which sort of a triangle without doubt would refer one who believed that these two straights AX, BX met one another at least at an infinite distance, although the angles at the transversal $PFHD$ were such as we have supposed.

Moreover, on account of the demonstrated common perpendicular KL, surely those two KX, LX cannot come together toward the part of the points X, since also (from a superposition easily understood) toward the other part also would meet at the same time the remaining and themselves unterminated KA, LB. Wherefore two straights AX, BX would enclose a space; which is contrary to the nature of the straight line.

But these things are later. For in the preceding I have never applied either the 16th or 17th of the first book of Euclid, except where clearly it treats of a triangle bounded on every side, as indeed I promised I would so take care to do *in Proemio ad Lectorem*.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from March Number.]

V. Let ABC be $\triangle$ right-angled at C. Draw FD perpendicular to AB, meeting either leg produced. There are thus four similar right triangles.

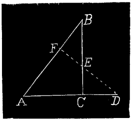


Fig. 5.

Letting $AC=b$, $AB=c$, $BC=A$, $CD=x$, $CE=y$, $AF=z$, $EB=a-y$, $FB=c-z$, $AD=b+x$, $FE=v$, $ED=w$, $FD=v+w$, we obtain the following proportions, with their resulting equations :

(1). $b : z :: c : b+x$. $\therefore b(b+x)=cz$........1.

(2). $b : z :: a : v+w$. $\therefore b(v+w)=az$......2.

(3). $c : b+x :: a : v+w$. $\therefore c(v+w)=a(b+x)$....................3.

(4). $b : y :: c : w$. $\therefore bw=cy$....................................4.

(5). $b : y :: a : x$. $\therefore bx=ay$....................................5.

(6). $c : w :: a : x$. $\therefore cx=aw$....................................6.

(7). $b : v :: c : a-y$. $\therefore b(a-y)=cv$............................7.

(8). $b : v :: a : c-z$. $\therefore b(c-z)=av$............................8.

(9). $c : a-y :: a : c-z$. $\therefore c(c-z)=a(a-y)$......................9.

(10). $z : y :: b+x : w$. $\therefore zw=y(b+x)$..........................10.

(11). $z : y :: v+w : x$. $\therefore xz=y(v+w)$..........................11.

(12). $b+x : w :: v+w : x$. $\therefore x(b+x)=w(v+w)$.................12.

(13). $z : v :: b+x : a-y$. $\therefore z(a-y)=v(b+x)$..................13.

(14). $z : v :: v+w : c-z$. $\therefore z(c-z)=v(v+w)$....................14.

(15). $b+x : a-y :: v+w : c-z$. $\therefore (c-z)(b+x)=(a-y)(v+w)$....15.

(16). $y : v :: w : a-y$. $\therefore y(a-y)=vw$..........................16.

(17). $y : v :: x : c-z$. $\therefore y(c-z)=vx$..........................17.

(18). $w : a-y :: x : c-z$. $\therefore w(c-z)=x(a-y)$..................18.

We are now to find combinations of the above equations from which the elements x, y, z, v, w, can be eliminated, thus leaving us the relation existing between a, b, and c.

It is evident that from no single equation, nor from any set of two equations, can the relation be determined.

There remains three possible cases of combinations to be considered :

1. When three of the elements x, y, z, v, w, are involved.
2. When four.
3. When five, or all.

First Case. Of this case there are $\frac{5.4.3}{|\underline{3}} = 10$ possible combinations of three unknown elements : v, w, x ; v, w, y ; and so on.

Before taking up these in detail, we note that by inspection of the proportions, it easily may be seen that the following eighteen sets of equations each comprise dependent equations :

1, 2, 3 ; 4, 5, 6 ; 7, 8, 9 ; 10, 11, 12 ; 13, 14, 15 ; 16, 17, 18 ; 1, 4, 10 ; 1, 7, 13 ; 2, 5, 11 ; 2, 8, 14 ; 3, 6, 12 ; 3, 9, 15 ; 4, 7, 16 ; 5, 18, 17 ; 6, 9, 18 ; 10, 13, 16 ; 11, 14, 17 ; 12, 15, 18.

Hence, in our search for possible combinations, all such must be rejected as contain any of these sets.

There are three equations involving v, w, x : 3, 6, 12. But this combination must be rejected for the reason just given. For the same reason, or because there is wanting a sufficient number of equations involving the three unknown elements, the other nine combinations must be rejected, except the combination x, y, z, which elements are involved in equations 1, 5, 9. If we eliminate x, y, z from these equations, we obtain the desired relation, $c^2 = a^2 + b^2$.

It should be observed, in passing, that future combinations including 1, 5, 9, must also be rejected.

Second Case. Of this case there are $\frac{5.4.3.2}{|\underline{4}} = 5$ possible combinations of four unknown elements ; and, besides, the exceptional combination, $v+w$, x, y, z, $v+w$ being regarded as a single unknown.

Before proceeding to investigate this case, it is necessary to call attention to sets of four dependent equations. Take, for example, the set 1, 2, 6, 12. From 1 and 2, 3 is obtained. But 3 with 6 and 12 gives a set of three dependent equations ; hence the set 1, 2, 6, 12 must be rejected. A little study of the eighteen sets given in Case 1, will disclose forty-five sets of four dependent equations.

The equations involving the unknown elements v, w, x, y, are 3, 4, 5, 6, 7, 12, 16. Out of these seven equations, there are $\frac{7.6.5.4}{|\underline{4}} = 35$ combinations, taking four at a time. Of these thirty-five sets, fourteen are to be rejected, for reasons previously stated. The remaining twenty-one sets, of which 7, 5, 4, 3, is a type, and to which the other twenty easily can be reduced, give, after the unknown elements have been eliminated, the desired relation between a, b, and c.

Similarly, we find twenty-one sets each of four equations, involving (v, w, x, z) and (v, w, y, z), and seventeen each involving (v, x, y, z), (w, x, y, z),

and $(v+w, x, y, z)$, thus making in all 114 proofs for this case.

THIRD CASE. Of this case, there are $\frac{18.17.16.15.14}{\lfloor 5}=8568$ sets of the eighteen equations, taking five at a time.

To determine how many of this number must be rejected, proceed as follows. Begin with the list of sets of dependent equations found in Case 1. Notice that there are $\frac{15.14}{\lfloor 2}=105$ sets of the eighteen equations taking five at a time, each containing equations 1, 2, 3; the same number containing equations 4, 5, 6; and so on, till we come to 1, 4, 10; for while there are 105 sets containing equations 1, 4, 10, three of them have already been counted out. So proceed, with the entire list of sets of dependent equations in Case 1, then with the set 1, 5, 9, following this with the sets of Case 2. We thus find that there are 3746 sets of five to be rejected, either because they contain sub-sets of dependent equations or sub-sets of equations from which the desired relation between a, b, c, is obtained.

One more class must be rejected: sets of five dependent equations. For example, 10, 9, 7, 6, 3, which is a type of all the others—72 in number—and from which the 72 can easily be deduced.

Deducting from 8568, 3746 + 73, we have remaining 4749 sets of five, from which can be derived the identity $c^2=a^2+b^2$.

$\therefore$ $1+114+4749=4864$, the number of proofs by this method.

EXAMPLES:

1. $cv+cw-ax=ab$ ....3.
   $bw=cy$ ....4.
   $aw=cx$ ....6.
   $cv+by=ab$ ....7.

4 in 6, $bx=ay$ ....5.

4, 5 and 7 in 3, $ab-by+\frac{c^2y}{b}-\frac{a^2y}{b}=ab$. $\therefore$ $c^2=a^2+b^2$.

2. $cz-bx=b^2$ ....1.
   $bv+bw-az=0$ ....2.
   $bw=cy$ ....4.
   $bx=ay$ ....5.
   $cv+by=ab$ ....7.

1 in 2, $cv+cw-ax=ab$ ....3.

4, 5, and 7 in 3, same as in 1st example.

VI. Let ABC be $\triangle$ right-angled at C. Produce AC to some point as D. Draw DF perpendicular to AB, produced, and meeting CB, produced.

Employing notation similar to that used in V., and proceeding somewhat in the same manner, we find that this method also yields a large number of proofs, in fact the same number that we found in V.

[To be Continued.]

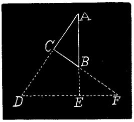


Fig. 6.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

56. Prophsed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A, B, and C can walk at the rate of $a=3$, $b=4$, and $c=5$ miles, per hour. They start from Washington, at $m=1$, $n=2$, and $p=3$ o'clock, P. M., respectively. When B overtakes A, he is ordered (by A) back to C. When will B and C meet? Suppose B had ordered A back to C, when would A and C meet? In case all three continue walking ahead, at what time will they meet?

Solution by P. S. BERG, Larimore, North Dakota.

Since B gains 1 mile in 1 hour on A, to gain 3 miles will require 3 hours, or it will be 5 o'clock and 12 miles from starting point when B and A meet. C has traveled 10 miles. Since B and C travel 9 miles in 1 hour, they will travel 2 miles in $\frac{2}{9}$ hour, hence they will meet at $5\frac{2}{9}$ o'clock. Since A and C travel 8 miles in 1 hour, they will travel 2 miles in $\frac{1}{4}$ hour, hence they will meet at $5\frac{1}{4}$ o'clock.

In case all three continue walking ahead, as stated above A and B will meet at 5 o'clock. Since C gains 2 miles on A in 1 hour, to gain 6 miles will require 3 hours. Hence they will meet at 6 o'clock. Since C gains 1 mile on B in 1 hour, to gain 4 miles will require 4 hours. Hence it will be 7 o'clock when they meet.

Also solved by *B. F. YANNEY* and *H. C. WILKS*.

57. Proposed by L. B. FRAKER, Weston, Ohio.

Suppose that in a meadow the grass is of uniform quality and growth and that 6 oxen or 10 colts could eat up 3 acres of the pasture in $\frac{18}{25}$ of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require $2\frac{6}{7}$ weeks longer than 660 sheep to eat up 9 acres.

In what time would an ox, a colt, and a sheep together eat up an acre of the pasture on the supposition that 589 sheep eat as much in a week as 6 oxen and 11 colts? By Arithimetic, if possible.—Hunter's Arithmetic. (Unsolved in *School Visitor.*)

Solution by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

1. By first condition, the eating capacity of a colt is to that of an ox as $6:10$.

$\therefore$ By last condition, the eating capacity of a colt is to that of a sheep as $589:21$.

$\therefore$ The eating capacity of a colt is to that of a sheep, an ox, and a colt together, as $1767:4775$.

2. $\therefore$ The first two conditions of the problem may be stated as follows:

10 colts could eat up 3 acres of the pasture in $\frac{18}{25}$ of the time in which 17 colts could eat up 6 acres, or 1400 colts would require $2\frac{6}{7}$ weeks longer than 1540 colts to eat up 589 acres.

3. Let $42u$ be the amount of grass consumed each week by a colt.

4. Suppose the time it takes 10 colts to eat up 3 acres is 18 weeks; then, the time it takes 17 colts to eat up 6 acres would be 25 weeks.

5. $\therefore$ $(10\times18\times42u)\div3=2520u$, total amount of pasture eaten from 1 acre in 18 weeks; and $(17\times25\times42u)\div6=2975u$, total amount of pasture eaten from 1 acre in 25 weeks.

6. $\therefore$ $(2975u-2520u)\div(25-18)=65u$, obviously the amount of growth on 1 acre in 1 week, and the same result that would be obtained whatever the time supposed in (4).

7. $\therefore$ $2520u-18\times65u=1350u$, amount of pasture originally on 1 acre.

8. $\therefore$ $(589\times1350u)\div(1400\times42u-589\times65u)=\frac{159030}{4103}$, the number of weeks it would take 1400 colts to eat of 589 acres of pasture; similarly, the time required for 1540 colts is found to be $\frac{159030}{5279}$ weeks. Now, the difference between these two numbers, $\frac{159030\times1176}{4103\times5279}$ weeks : $2\frac{6}{7}$ weeks, the true difference :: 18 weeks, the supposed time : the true time.

9. $\therefore$ Since the only number that needs correcting, to enable us to complete the solution, is $1350u$, the amount of pasture originally on 1 acre, the time required for an ox, a colt, and a sheep together to eat up 1 acre, is

$$\left(\frac{20}{7}\times\frac{4103\times5279}{159030\times1176}\times1350u\right)\div\left(\frac{4775}{1767}\times42u-65u\right)=9\frac{2482499}{11757354}\text{weeks. Answer.}$$

H. C. Wilkes gets 142.25 + days.

NOTE. This problem appeared a few years ago in the *School Visitor*. With no little difficulty, we obtained a solution by Algebra. The solution was not published because of the difficult composition. It is strange that such a problem should appear in an arithmetic which is to be used by boys and girls 16 years old and upwards. EDITOR.

PROBLEMS.

58. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Two men, A and B, in Boston, hire a carriage for \$25, to go to Concord, N. H., and back, the distance being 72 miles, with the privilege of taking in three more persons. Having gone 20 miles, they take in C; at Concord, they take in D; and when within 30 miles of Boston, they take in E. How much shall each man pay? [From *Greenleaf's National Arithmetic*.]

59. Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.

A broker charges me $1\frac{1}{2}$ per cent. brokerage for buying some uncurrent bank bills at 20 per cent. discount. Of these bills 4 of \$50. each become worthless, but the remainder I dispose of at par, and make by the operation \$364. What was the face amount? [Which answer is correct, \$3000, or $\$3048\frac{24}{37}$?]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

56. Proposed by CHAS. E. MYERS, Canton, Ohio, and Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

(a) How much can be paid for a bond, bearing 5 per cent. interest, and having ten years to run, so as to realize 3 per cent. on the investment? (b) At what price must the government sell 5 per cent. \$100 bonds to run ten years, interest payable annually, to make them the same to the buyer as 3 per cent. bonds at par, to run ten years, interest payable annually, provided the buyer can invest all interest received at 4 per cent. interest, payable annually?

Solution by J. K. ELWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Let x=price, a=face, n=number of periods, R=rate bond bears, r=rate to be realized, r' =rate on interest.

The interest on bond is an annuity at compound interest whose final value $=\frac{Ra}{r'}[(1+r')^n-1]$, which added to the *face* value of bond must equal the compound amount of the *price* for n periods, or $x(1+r)^n$.

$\therefore x=\dfrac{a+Ra[(1+r')^n-1]}{(1+r)^n}$. For (a), $a=100$, $n=10$, $R=.05$, $r=.03$, $r'=.03$.

$$\therefore\ x=\frac{100+\frac{5}{.03}(1.03^{10}-1)}{1.03^{10}}=\$117.0604.$$

For (b), $a=100$, $n=10$, $R=.05$, $r=.03$, $r'=.04$.

$$\therefore\ x=\frac{100+\frac{5}{.04}(1.04^{10}-1)}{1.03^{10}}=\$119.0777.$$

If in (a) interest were payable semi-annually, we should have $a=100$, $n=20$, $R=.025$, $r=.015$, $r'=.015$, and $x=\$117.168+$, or \$117.17 as given in the tables of bond values used by brokers and bankers.

Also solved by *E. W. MORRELL*, *B. F. YANCEY* and *G. B. M. ZERR*. Prof. Morrell obtained as results \$118.356 and \$117.661; and Proposer, to last part, \$117.60.

57. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

Find the quotient of

$$\begin{vmatrix} (s-a_1)^2 & a_1^2 & a_1^2\ldots\ldots a_1^2 \\ a_2^2 & (s-a_2)^2 & a_2^2\ldots\ldots a_2^2 \\ a_3^2 & a_3^2 & (s-a_3)^2\ldots\ldots a_3^2 \\ \ldots & \ldots & \ldots \\ a_n^2 & a_n^2 & a_n^2\ldots . s-a_n^2 \end{vmatrix} \div \begin{vmatrix} s-a_1 & a_1 & a_1\ldots\ldots\ldots\ldots a_1 \\ a_2 & s-a_2 & a_2\ldots\ldots\ldots\ldots a_2 \\ a_3 & a_3 & s-a_3\ldots\ldots\ldots a_3 \\ \ldots & \ldots & \ldots \\ a_n & a_n & a_n\ldots\ldots\ldots s-a_n \end{vmatrix}$$

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let Q=the quotient and as we can exchange row for column without altering the value, we get

$$Q=\begin{vmatrix} (s-a_1)^2 & a_2^2 & a_3^2\ldots . a_n^2 \\ a_1^2 & (s-a_2)^2 & a_3^2\ldots . a_n^2 \\ a_1^2 & a_2^2 & (s-a_3)^2\ldots . a_n^2 \\ \ldots & \ldots & \ldots \\ a_1^2 & a_2^2 & a_3^2.(s-a_n)^2 \end{vmatrix} \div \begin{vmatrix} s-a_1 & a_2 & a_3\ldots\ldots\ldots a_n \\ a_1 & s-a_2 & a_3\ldots\ldots\ldots a_n \\ a_1 & a_2 & s-a_3\ldots\ldots\ldots a_n \\ \ldots & \ldots & \ldots \\ a_1 & a_2 & a_3\ldots . s-a_n \end{vmatrix}$$

All the elements in the i^{th} column of the numerator being a_i^2, of the denominator a_i, except in the i^{th} row which is $(s-a_i)^2$ for numerator, and $s-a_i$ for denominator. Hence, we have

$$Q=\begin{vmatrix} 1, & 0, & 0, & 0,\ldots\ldots \\ 1, & (s-a_1)^2, & a_2^2, & a_3^2, \ \ldots \\ 1, & a_1^2, & (s-a_2^2,) & a_3^2, \ \ldots \\ 1, & a_1^2, & a_2^2, & (s-a_3)^2, \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \div \begin{vmatrix} 1, & 0, & 0, & 0,\ldots\ldots \\ 1, & s-a_1, & a_2, & a_3,\ldots\ldots \\ 1, & a_1, & s-a_2, & a_3,\ldots\ldots \\ 1, & a_1, & a_2, & s-a_3,\ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

Multiply first column of numerator by a_i^2, of the denominator by a_i and subtract from the i^{th} column; do this for each column and the value is unaltered.

$$\therefore\ Q=\begin{vmatrix} 1, & -a_1^2, & -a_2^2, & -a_3^2, \ldots \\ 1, & s(s-2a_1), & 0, & 0, \ldots\ldots \\ 1, & 0, & s(s-2a_2), & 0, \ldots\ldots \\ 1, & 0, & 0, & s(s-2a_2), \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \div \begin{vmatrix} 1, & -a_1, & -a_2, & -a_3, \ldots \\ 1, & s-2a_1, & 0, & 0, \ldots \\ 1, & 0, & s-2a_2, & 0, \ldots \\ 1, & 0, & 0, & s-2a_3, \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

Let $u=(s-2a_1)(s-2a_2)(s-2a_3)\ldots\ldots(s-2a_n)$.

$$\sum\frac{a_i^2}{s-2a_i}=\frac{a_1^2}{s-2a_1}+\frac{a_2^2}{s-2a_2}+\frac{a_3^2}{s-2a_3}+\ldots\ldots$$

$$\therefore\ Q=\frac{s^{n-1}u\left\{s+\sum\frac{a_i^2}{s-2a_i}\right\}}{u\left\{1+\sum\frac{a_i}{s-2a_i}\right\}}=\frac{s^{n-1}\left\{s+\sum\frac{a_i^2}{s-2a_i}\right\}}{\left\{1+\sum\frac{a_i}{s-2a_i}\right\}}.$$

ERRATA. On page 52 of last issue, line 3 from bottom, read = before $\frac{1}{c}$, and in the denominator read $\sqrt{a^2-x^2}$ for "$\sqrt{a^2+x^2}$"; on page 53, line 15, extend the radical sign over a^2-x^2 and b^2-x^2, in the numerators.

PROBLEMS.

64. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Solve the equations:

$$a^2x=(2x^2-a^2)\sqrt{x^2+y^2}\ \ldots\ldots\ (1).$$

$$b^2y=(2y^2-b^2)\sqrt{x^2+y^2}\ \ldots\ldots(2).$$

65. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Prove that $\cos\frac{n\pi}{7}+\cos\frac{3n\pi}{7}+\cos\frac{5n\pi}{7}=\frac{1}{2}$ or $-\frac{1}{2}$, according as n is *odd* or *even*.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

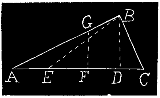
50. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Divide a triangle into the ratio of m to n by a line perpendicular to the base.

Solution by J. C. GREGG, Superintendent of Schools, Brazil, Indiana; E. W. MORRELL, Professor of Mathematics, Montpelier Seminary, Montpelier, Vermont; and the PROPOSER.

Let ABC be the triangle. Draw the altitude BD. Divide the base AC at E so that $AE:EC=m:n$. Draw the line BE.

Then $\triangle ABE : \triangle EBC = AE : EC = m:n$.....(1).



Take AF a mean proportional between AE and AD, then draw GF parallel to BD.

Then $\triangle AFG : \triangle ADB = AF^2 : AD^2$.

But $AF^2 = AE\times AD$.

$\therefore \triangle AFG : \triangle ADB = AE\times AD : AD^2 = AE : AD = \triangle ABE : \triangle ADB$.

$\therefore \triangle AFG = \triangle ABE$ and $\triangle EBC = FGBC$.

Hence, using in (1), we have $\triangle AFG : FGBC = m:n$. Q. E. D.

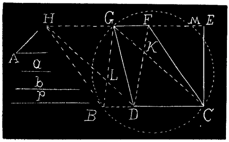
Also solved in various ways by *G. B. M. ZERR, B. F. YANNEY, J. SCHEFFER, A. H. BELL, F. R. HONEY, O. W. ANTHONY, H. J. GAERTNER, G. I. HOPKINS, J. M. COLAW, J. O. MAHONEY.*

51. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Construct a tropezoid, given the bases, the altitude, and the angle formed by the intersection of the diagonals.

Solution by J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee; FREDERICK R. HONEY, A. B., New Haven, Connecticut; J. SCHEFFER, Hagerstown, Maryland; B. F. SINE, Principal of High School, Rock Enon Springs, Virginia; and PROPOSER.

Let a and b be the bases, p the perpendiculars, and A the angle between the diagonals.



Take $BC=a+b$ and describe upon BC a segment to contain an angle=to A. The problem is possible when p is less than the greater segment of the diameter perpendicular to BC. Take $CE=p$ and perpendicular to BC. Draw EH parallel to BC cutting the circle in M and G. Draw BG and GC. Also draw DF parallel to BG and DH parallel to GC.

Then is $DCFG$ or $BDGH$ the required trapezoid. For $BD=GF=b$, $DC=HG=a$, $\angle DKC=\angle BLD=\angle BGC=A$, and $CE=p$. By treating the point m as we did G we get two other trapezoids answering all conditions.

This problem was solved in a similar manner by *COOPER D. SCHMITT, A. H. BELL, J. SCHEFFER, B. F. SINE, J. M. COLAW, P. S. BERG, O. W. ANTHONY, E. W. MORRELL, J. C. GREGG,* and *H. J. GAERTNER.*

PROBLEMS.

56. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of the centers of the isogonal transformations of all the diameters of the circumcircle of any triangle is the nine-points circle. *Brocard.*

57. Proposed by J. OWEN MAHONEY, B. E., Graduate Fellow and Assistant in Mathematics, Vanderbilt University, Nashville, Tennessee.

Show that pairs of points, on a straight line may be so related harmonically that a pair of real points will be harmonic with regard to a pair of imaginary points, and by this means prove that there are an indefinite number of conjugate pairs of imaginary points on a real line.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

45. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A fly starts from a point in the circumference of a table, 3 feet in diameter, and travels uniformly along the diameter to a point in the circumference of the table directly opposite the starting point. The table moves uniformly to the right about a center axis in such manner that it makes one complete revolution while the fly passes over its diameter. Find the absolute path described by the fly and the ratio of rates of movement of the table and the fly.

I. Solution by the PROPOSER.

The curve described by the fly is the spiral of Archimedes. Its equation is $r=a\theta$. $S=\int_0^\pi\left(\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\right)d\theta,=\frac{a\pi\sqrt{1+\pi^2}}{2}+\frac{a}{2}\log(\pi+\sqrt{1+\pi^2})$.

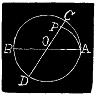
Hence, $2S$, or the absolute path described by the fly, is 63.994+ inches.

If we take the Naperian logarithm of $(\pi+\sqrt{1+\pi^2})$ the result is 69.6+ inches.

The ratio of rates$=\frac{2\pi r}{2r}=\pi$. The ratio of rates in *space*$=\frac{2\pi r}{63.994}=1.76+$.

II. Solution by G. B. M. ZERR, Ph. D., Professor of Mathematics in Texarkana College, Texarkana, Arkansas-Texas; and Professor J. SCHEFFER, A. M., Hagerstown, Maryland.

Let P be the position of the fly when A has moved to C, and let A move m times as fast as P. Let $OA=r$, $OP=\rho$, $\angle COA=\theta$.
Then $mPC=AC$. $\therefore m(r-\rho)=r\theta$.



$\therefore \rho=\frac{r(m-\theta)}{m}=\frac{r(\pi-\theta)}{\pi}$, since $m=\pi$. This is the equation to the fly's path.

$$\therefore S=\int_0^{2\pi}\frac{r}{\pi}\sqrt{1+(\pi-\theta)^2}\,d\theta$$

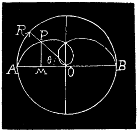
$$=r\sqrt{1+\pi^2}+\frac{r}{\pi}\log(\pi+\sqrt{1+\pi^2}).$$

$$\therefore S=\tfrac{3}{2}\left\{\sqrt{1+\pi^2}+\frac{1}{\pi}\log(\pi+\sqrt{1+\pi^2})\right\}=5.835 \text{ feet.}$$

$$\frac{3\pi}{S}=\frac{1885}{1167}=\frac{13}{9}\text{ nearly.}$$

III. Solution by Prof. J. M. BANDY, A. M., Old Trinity College, North Carolina, and J. C. GREGG, Superintendent of City Schools, Brazil, Indiana.

Let (ρ, θ) denote the co-ordinates of P, and since AR and RP are in a constant ratio, ρ and θ are in the same ratio, which denote by c.



Hence, $\theta=-\rho c$ [Archimedean spiral]......(1).
By theory of curves,

$$S=\int\left(\rho^2+\frac{d\rho^2}{d\theta^2}\right)^{\frac{1}{2}}d\theta\text{..................(2).}$$

From (1), $\frac{d\rho^2}{d\theta^2}=\frac{1}{c^2}$, and $\rho=\frac{\theta^2}{c^2}$. Substituting these values in (2), $S=\frac{1}{c}\int_0^{2\pi}(1+\theta^2)^{\frac{1}{2}}\,d\theta$..............................(3).

Integrating (3) by formula for reducing $p=1/2$,

$$S=\left[\frac{\theta(1+\theta^2)^{\frac{1}{2}}}{2c}\right]_0^{2\pi}+\frac{1}{2c}\log\left[\theta+\sqrt{1+\theta^2}\right]_0^{2\pi}\text{....................(4).}$$

But $c=\frac{\text{arc } AR}{RP}=\frac{\text{circumference}}{\text{diameter}}=\frac{\pi}{1}$. Substituting in (4), and reducing, $S=6.4533+$ feet.

The movement of the fly in its path is the resultant of the motion of the fly along the diameter and the motion of the table to the right about its axis. The rate of motion of the fly in its path is variable, and is measured at any instant by the measuring circle given by any particular value of ρ. So that the ratio of the motion of the table to that of the fly can be found for any particular value of ρ.

Also solved by *O. W. ANTHONY* and *E. L. SHERWOOD*. Prof. Sherwood's solution will be published under problem No. 50.

46. Proposed by H. C. WHITAKER, M. E., Sc. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

There are four points, A, B, C, and D in space. Point D remains fixed with its co-ordinates (1, 2, 2) feet. At a given time A is at (2, 3, 4) feet, is moving in a straight line at the rate of 3 feet per minute, and has passed through (5, 9, 10) feet; B is at (1, 4, 2) feet, moves in a straight line at the rate of 7 feet per minute, and will pass through (—2, 2, 8) feet; C is at the origin and moves along the axis of X in the direction of x positive at the rate of 6 feet per minute.

The motion of the points being continuous before and after the given time, required the times when the volume of the tetrahedron whose edges are the lines joining these points will be 108 cubic inches.

Solution by the PROPOSER.

The length of a base edge [from (x_1, y_1, z_1) to (x_2, y_2, z_2)] is well known to be

$$\sqrt{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}^2+\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}^2+\begin{vmatrix} z_1 & 1 \\ z_2 & 1 \end{vmatrix}^2}$$

Finding the distance from $(x_3y_3z_3)$ to this edge, multiplying this distance by the length of the edge just given, the area of the base is

$$\frac{1}{\lfloor 2}\sqrt{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2+\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}^2+\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2}$$

Finding the distance from $(x_4y_4z_4)$ to this base, multiply this distance by the area of the base just given, the volume of the tetrahedron is found to be *

$$\frac{1}{\lfloor 3}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

[*The extension of each of these values in n dimentioned co-ordinates is obvious, as is also the sursolidity of a figure in four dimensioned space bounded by five tetrahedra; and so on.]

Substituting the given values of the co-ordinates, we have

$$\frac{1}{6}\begin{vmatrix} 2-t & 3-2t & 4-2t & 1 \\ 1-3t & 4-2t & 2+6t & 1 \\ 6t & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 \end{vmatrix}=V.$$

This reduces to $16t^3-14t^2+2t=\pm.0625$; whence by solving, $t=-.026$, $.045$, $.125$, $.217$, $.684$, and $.705$ seconds, respectively.

Also solved by *J. SCHEFFER* and *G. B. M. ZERR*.

PROBLEMS.

53. Proposd by O. D. SMITH, A. M., Professor of Mathematics, Alabama Polytechnic Institute, Auburn, Alabama.

Solve the differential equation, $dy/dx=y(x-y)/x(x+y)$; and show that $x=y\log(x\ y)$.

54. Proposed by Prof. J. SCHEFFER, A. M., Hagerstown, Maryland.

A certain solid has a square, side$=a$, for its base, and all parallel sections are squares, the two sections through the middle points of the opposite sides of the square are semi-circles, however. Find surface, volume, and the centers of gravity of each.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

PLAYFAIR'S PSEUDO-PROOF OF THE ANGLE-SUM.

BY GEORGE BRUCE HALSTED.

The living person who has most capital invested in Playfair's fallacious demonstration reproduced in the March number of THE AMERICAN MATHEMATICAL MONTHLY, pp. 77—79, is Professor George C. Edwards of the University of California, who unfortunately gives it as the basis for his treatment of parallels in §16 of his Elements of Geometry, Macmillan, 1895.

His §16 is Playfair's Proposition I "All the exterior angles equal four right angles," with Playfair's fallacious proof. Then his §17 is "THEOREM. If two straight lines make equal angles with a third straight line intersecting them, they will make equal angles with *any* straight line intersecting them," in proving which he twice cites §16. Then as Exercise 1 under §17 he has "Establish the theorem when the fourth line passes through B." But this very special case of his §17 he assumes in his §16, thus making his treatment of parallels a simple

argumentum in circulo. I wrote this to Professor Edwards and he wrote in answer what clearly seemed an explicit acknowledgment of it. But it was so unlike a paradoxer to acknowledge a fallacy, that in wonder I wrote again, "You mean to state that in your proof of the theorem §16 of your book, you do assume (without stating the assumption) your Exercise 1 under your §17. Am I right in this understanding of your letter?" And strange as it may seem he wrote March 7th, 1896, "You are practically right in your understanding of my letter of February 22nd."

I have given three different exposures of Playfair's fallacy in the fourth edition of my Bolyai pp. 65—71.

THEORY AND PRACTICE COMBINED.

BY WARREN HOLDEN, GIRARD COLLEGE, PHILADELPHIA, PENNSYLVANIA.

Common experience, applied to Mechanical and Engineering problems, has always been in harmony with the principles of Euclidean Geometry. With the overthrow of these principles we might expect chaos to come again. And if Mathematics has not yet demonstrated all of these principles, so much the worse for Mathematics. Let its Professors try again. Their failure in any particular case does not establish the opposite.

Abstract studies in Philosophy, unmodified and unillustrated by human experience, have often led to bewildering vagaries. Does not a similar fate, from corresponding causes, impend over Non-Euclidean Geometry? Theory and practice should go hand in hand.

All mathematical instruments in use, whether in the department of Mechanics, Physics or Engineering, are constructed upon the basis of Euclidean Geometry. Where are the instruments of precision which serve to illustrate and apply the principles of Non-Euclidean Geometry?

QUERIES.

1. Please give me address of publishing house that publishes the most reliable works on How to Calculate Timber on the Stump, also names of most reliable works on same.

JOHN BRIDGES.

EDITORIALS.

Prof. C. A. Waldo is now Professor of Mathematics in Purdue University, Lafayette, Indiana.

Science, March 27, contains an able article, The Essence of Number, by Dr. George Bruce Halsted.

Prof. J. A. Calderhead has been elected Professor of Mathematics in the Curry University, Pittsburg, Pennsylvania.

Dr. Byerly's Fourier's Series and Spherical Harmonics, we are informed by the Publishers, is gaining an international reputation.

Dr. E. H. Moore has been promoted to Head Professor of Mathematics in the University of Chicago. This is a merited recognition.

Professor J. J. Sylvester, formerly of the Johns Hopkins University, has just been made a Foreign Member of the Turin Royal Academy of Science.

Our subscribers will do us a kindness by sending us the names of persons who are likely to subscribe for the MONTHLY, as we would be pleased to send such persons sample copies.

A few of our former subscribers who are in arrears have asked us to discontinue the MONTHLY to their address. In no case will we discontinue to send the MONTHLY until the amount due us is paid.

Mr. W. J. C. Miller, who is editor of the Mathematical Department of the *Educational Times*, London, England, says, "THE AMERICAN MATHEMATICAL MONTHLY is one of the best magazines that I receive." Mr. Miller has edited the Mathematical Department of the *Educational Times* for over 30 years.

M. A. Gruber, of Washington, D. C., writes: You will please find enclosed a Money Order of $3.00 as my subscription to THE AMERICAN MATHEMATICAL MONTHLY for 1896. It is a magazine worthy of long life; if the additional mite is any assistance in putting it upon a paying basis, I shall always remain among your best friends.

We have on hand a few bound copies of Volumes I and II which we will sell at $2.75 each. By special arrangements with the binders we can have volumes of the MONTHLY bound for 75 cents. If any of our subscribers wish to avail themselves of this opportunity to have their volumes of the MONTHLY bound, they may send them to B. F. Finkel, Springfield, Mo.

Philadelphia Summer Meeting will hold its fourth session, July 6—31, 1896, in the buildings of the University of Pennsylvania, under the auspices of the American Society for the Extension of University Teaching. Department E —Mathematics: I. Methods of Teaching Mathematics; II. Plane and Solid Geometry; III. Algebra (Elementary Course); IV. Algebra (Advanced Course); V. Trigonometry; VI. Analytical Geometry; VII. Differential and Integral Calculus; VIII. Theory of Equations and Determinants; IX. Differential Equations; X. Theory of Functions.

The lecturers are I. J. Schwatt, Ph. D., and G. H. Hallett, M. A., of the University of Pennsylvania. On Wednesday evening, July 8, Dr. Schwatt will deliver to the students of all departments of the Summer Meeting an address on the Philosophy and Utility of the Calculus.

We are sorry to announce the death of one of our valued contributors, T. P. Stowell, of Rochester, N. Y., which occurred February 29th, 1896. Mr.

Stowell's name has been closely associated with nearly all the mathematical journals published in this country within the last fifty years. The following sketch is taken from *The Union and Advertiser*, Rochester, New York:

Thomas P. Stowell, of No. 29 Atkinson street, died Saturday at the home of the family, aged 77 years. Mr. Stowell, who had resided in the city since April 1, 1864, at the residence now occupied by the family, was born September 5, 1819, and was widely known, respected and esteemed, not only in Rochester but throughout the entire country. He graduated from the well-known Hallowell University of Virginia, and was considered one of the ablest mathematicians in the United States. He retired from business in 1895, in the enjoyment of robust health, having apparently the strength and certainly the appearance of a middle-aged man.

Mr. Stowell had been a member of St. Luke's Church during the entire period of his residence in Rochester. He leaves a wife and five children, Miss Anna Stowell, Miss M. Louise Stowell, Dr. Henry F. Stowell, and C. L. Stowell, all of this city, and Charles F. Stowell of Albany.

BOOKS AND PERIODICALS.

Syllabus of Geometry. By G. A. Wentworth, A. M., Author of a Series of Text-books in Mathematics. Pamphlet form. 50 pages. Boston and Chicago: Ginn & Co.

This pamphlet contains the enunciations of the propositions and corollaries of the author's text-book in Geometry, numbered as they are in the text-book. B. F. F.

Rational Mathematics. By Charles De Medici.

Under the above title the author is publishing a work—The New Geometry and Commensurational Arithmetic—which is divided into three sections: A, B, C. In Section A, Part I, the first principles and primary elements of Geometry are taught; Part II. First principles of Commensuration, founded on the Natural Division and Inherent Dimensions of Geometric Elements are taught; Part III. Classification of Geometric Figures and Forms. Section B, Geometry Study and Practice. The work is published by A. Lovell & Co., New York. B. F. F.

Elementary Treatise on Electricity and Magnetism Founded on Joubert's Traité Élémentaire D'Éléctricité. By G. C. Foster, F. R. S., Quain Professor of Physics in University College, London, and E. Atkinson, Ph. D., formerly Professor of Experimental Science in the Staff College. 8vo. Cloth, 552 pp. Introduction price, $1.80. New York: Longmans, Green & Co.

This treatise on Electricity and Magnetism is confined to facts, hypotheses being studiously avoided. The treatment of each subject is clear, simple, direct, and exhaustive. Whenever necessary, the higher mathematics are used in compututions and the establishment of electrical laws. It is the best treatise on Electricity and Magnetism that we have yet seen and we heartily commend it to any person desiring a good work on these important subjects. B. F. F.

Elementary Algebra. By J. A. Gillett, Professor in the New York Normal College. 8vo. Half Leather Back. 412 pp. New York: Henry Holt & Co.

Among other commendable features of this book may be mentioned, (1) the prominence given to problems and the consequent introduction of the equation, (2) the attention given to negative quantities, (3) the attention given to the formal laws of Algebra,—the Commutative, the Associative and the Distributive laws, and (4) the simplicity, clearness, and logical arrangement of the matter. The book is beautifully printed and handsomely bound, and presents a most attractive appearance.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single number 25 cents. The Review of Reviews Co., New York City.

The *Review of Reviews* is almost indispensable to the general reader who wishes to keep abreast of the rapidly developing international questions of the day. In the April number there is a full and able editorial discussion of the complicated African situation, which is described as "the drama of 'Europe in Africa.'" The mixed interests and motives of England, Russia, Italy and France in the Dark Continent are clearly set forth. Russia's general attitude toward the European powers is also discussed, and the editor comments briefly on America's relations with Spain, our interests in the Cuban revolution, and the present status of the Venezuelan boundary dispute. In addition to this editorial treatment (in the department entitled "The Progress of the World") the *Review* presents a remarkably complete survey of the Cuban situation by Murat Halstead, a summary of the best current thought in England on the subject of international arbitration, and a vivid account of the relief work now going on in Armenia. In short the *Review of Reviews* records a month's activities in both hemispheres.

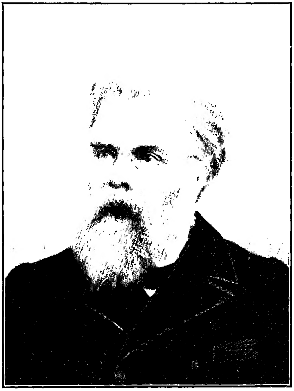
April Monthly Magazine Number of the Outlook. Price, $1.00 per year in advance. The Outlook Company, 13 Astor Place, New York.

In the April Magazine Number of *The Outlook* there will appear an article on William H. Prescott, by Kenyon West. It will be in commemoration of the centenary of the great American historian, who was born May 4, 1796. The article will be enriched by uumerous portraits and other illustrations contirbuted from the private collection of members of the Prescott family, who have been interested in Kenyon West's tribute to Prescott. Among these are Mr. Arthur Dexter, of Boston, the nephew of the historian; Mrs. Roger Wolcott, Prescotts grand-daughter, who lives also in Boston; and Mr. Linzee Prescott of Greenwich, Conn., who is the son of Prescott's eldest son.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson, New York.

The April *Cosmopolitan* contains the following: A word about Golf, Golfers, and Golf-links in England and Scotland, by Price Collier; Vicissitudes of the Dead, by Eleanor Lewis; Development of the Overland Mail Service, by Thomas L. James; The Lyceum, by James B. Pond; Mrs. Cliff's Yacht, by Frank R. Stockton; The Bargain of Faust (Poem) by Alice W. Rollins; Hilda Stafford, by Beatrice Harraden. Each of these articles are beautifully illustrated.

The following periodicals have been received: Journal de Mathématiques Élémentaires, (15 Mars 1896); American Journal of Mathematics, (April, 1896); The Mathematical Gazette, (October, 1895); L'Intermédiaire des Mathématiciens, (Mars, 1896); El Progreso Matemático, (Tomo V. Ano 1895); Notes and Queries, (April, 1896); The Kansas University Quarterly, (January, 1896); Popular Astronomy, (June, 1895); The Monist, (April, 1896); Bulletin of the American Mathematical Society, (March, 1896); The Educational Times, (March, 1896).



EMILE-MICHEL-HYACINTHE LEMOINE.